# Semi Unit Graphs in Quotient Semirings

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## 1 Semi Unit Graphs in Quotient Semirings

**Abstract**

The main purpose of this paper is to develop some new graphs of semi unit elements in Quotient Semirings, we further study the characteristics of these graphs by using different conditions on Quotient Semirings.

## 2 Introduction and Preliminaries

The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many mathematicians with different angles. The concepts of the graph to the element (zero-divisor) of a ring was introduced by Beck in [17] when discussing the coloring of a commutative ring. The zero-divisor graph of a commutative ring has been studied extensively by several authors like D. F. Anderson , P. S. Livingston and Shahabaddin Ebrahimi Atani,e.g in [8,9,11]. But S.E Atani and his colleagues establish the propertes of zero divisor graphs linked with semirings .He also introduced unit graph and some other useful graphs in semirings in [9,11].We developed the Semi Unit graphs and studied its charecteristics in Semirings.In this paper, We focus our work on Semiunit Graphs in Quotient semirings and study characteristics of these graphs.
For the sake of completeness, we state some definitions and notations used throughout. A non-empty set S is called semi ring, if it is semi group under two binary operations addition and multiplication as well ,and these two binary operations are linked with distributive law. Some Mathematicians includes the additive identity zero also in semi ring. A Semiring S is commutative if ab = ba for all $a, b \in \ S$ and this commutative semiring S is commutative with identity if there exist $1 \in S$ such that 1.a = a.1 for all

$a \in$ S . In this paper we utilized the semi ring as commutative semi ring with non-zero identity else mentioned otherwise. A non-zero element a of S is said to be unit in S if there exist a non-zero element a of S is said to be unit in S if there exist $0 \neq r \in S$ , such that r.a= a.r = 1.
Let $S \neq \phi$ , be a commutative semi ring with non-zero identity. A non-zero element a of S is said to be semi unit in S if there exists, $r, s \epsilon S$ such that $1 + r.a = s.a$ . The set of all semi units of S will by denoted by $S_u$ and the set of all non-semi units will be denoted by $N_u$ in this paper.Every unit is a semi unit as by taking r=0.In a ring every semi unit is a unit.A non-empty subset I of S is called an ideal of S if$a, b \in I$ and $r \in S$ then $a + b \in I$ and $r.a \in I$ .The Radical of an ideal of semi ring S is given by Rad ( I) = { $a \epsilon S$ ; there exists a positive integer n such that $a^n \epsilon I$ }. A prime ideal of semi ring S is a proper ideal P of S in which $x \epsilon P$ *or* $y \epsilon P$ *whenever* $x.y \epsilon P$. A proper ideal M of semi ring S is said to be maximal , if M is an ideal in S such that $M \subsetneq J$, then J=R. Every Maximal ideal is prime if $S$ is commutative semiring with unity [6]. A Subtractive ideal (k-ideal) $K$ is an ideal (*may not proper*) such that if $x, x + y \in K$ than $y \in K$. 0 is a k-ideal of semi ring S. A k-ideal which is also maximal ideal is called Maximal k-ideal .if P is maximal k-ideal of S if and only if S/P is a semi field [6]. Let P be an ideal of a semi ring S , P is a prime k-ideal of S if and only if S/P is a semi domain. Let S be a semi ring with non-zero identity. S is said to be a local semi ring if and only if S has a unique maximal k-ideal. The k-Closure cl(I) of ideal I is defined by cl (I) ={ $a \epsilon S$ ; a + c = d for $c, d \epsilon I$ } is an ideal of S.
An ideal I of semiring S is called a partitioning ideal(Q-ideal) if there exists a subset Q of S such that: S =∪ {q +I : q ∈Q } . If $q_1, q_2 \epsilon Q$ then $(q_1 + I) \bigcap (q_2 + I) \neq \phi$ if and only if $q_1 = q_2$. If I is a partitioning ideal(Q-ideal) of a semiring S then I is a subtractive ideal (k-ideal) of S ,by (lemma 2[16]) .
A graph $\xi$ consists of a vertex set $V(\xi)$ and an edge set $E(\xi)$, where an edge is an unordered pair of distinct vertices of $\xi$ .The graph with no edges is the null graph. The number of vertices in a graph $\xi$ is called its order, and number of edges are its size.A graph with p- vertices and q- edges is said to be a (p, q) graph. Therefore (1, 0) is called the trivial graph and (p, 0) is called an empty or null or void graph. A graph with p- vertices and q- edges is said to be a (p, q) graph.

A graph with no loop or multiple edges is called a simple graph. The degree of a vertex v in a graph , denoted by d(v), is the number of edges of incident with v, each loop counting as two edges.A graph is regular if degree of every vertex are same. If every vertex is adjacent with n other vertices

than this graph is called n-regular graph.The distance between two vertices of a graph $\xi$ is the number of edges among the shortest path of these two vertices . If there is no path then the distance is taken as infinity. The diameter of a graph $\xi$ is the greatest distance between two vertices of $\xi$ .
There are some special families of graphs as complete graph is a simple graph in which any two vertices are adjacent.A graph is connected if there is a path between any two vertices.A graph is totally disconnected if every two vertices are non-adjacent. A complete graph is closed if each pair of vertices is joined by every edge (loops included) that if there is loop then every vertex must have loop. If not than no vertex has loop. A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y .If $\xi$ is simple and every vertex in X is joined to every vertex in Y, then $\xi$ is called a complete bipartite graph. We write $K_{m,n}$ to mean a complete bipartite graph with m vertices in one set and n in the other.A star is a complete bipartite graph $\xi$ with $|X| = 1 or |Y| = 1$ .
Let $\xi$ (V, E) be a simple graph. The complement $\xi^c$ of $\xi$ is the simple graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of $\xi$ . A cycle that passes through every vertex in a graph is called a Hamilton cycle and a graph with such a cycle is called Hamiltonian.An isomorphism between two simple graphs X and Y as a bijection $\theta :$ V (X) $\longrightarrow$ V (Y) which preserves adjacency .

## 2.1 Theorem

Let S be a commutative semi ring with non-zero identity and suppose that J(S) denotes the Jacobson radical of S. If x,y$\in S$ and If $x+J(S)$ and $y+J(S)$ are adjacent in $\xi(\dfrac{S}{J(S)})$ , then every element of $x + J(S)$ is adjacent to every element of $y + J(S)$ in $\xi(S)$.

**Proof** Suppose $x + J(S)$ and $y + J(S)$ are adjacent in $\xi(\dfrac{S}{J(S)})$ . So, there exist a semi unit u $\in S_u$ [Note: If u is semi unit in S, then u + I is semi unit in $\dfrac{S}{I}$ ]. In such a way $(x + J) \oplus (y + J) = u + J$,

$$\Rightarrow x + y + J \subseteq u + J \qquad (1)$$

Now, suppose that $a \in x + J, b \in y + J$. We may write $a = x + j_1$ and $b = y + j_2$, where $j_1, j_2 \in J$. Suppose on contrary, that a + b is non-semi unit then $(a + b)S \subseteq cl(a + b)S \subseteq M$ ,where M is some k-maximal ideal in S. Also

$$1 \in S \Rightarrow a + b \in M \qquad (2)$$

.
Here $a+b=x+j_1+y+j_2 \Longrightarrow a+b=x+y+j_3=u+j_4$ (by (1)) . From (1) and (2) , a + b = u + $j_4 \in M$ We know that J(S) $\subseteq M$ so $j_4 \in M \Rightarrow u \in M$, which is contradiction .
Hence a + b is semi unit that they are adjacent. ■

A clique in a graph $\xi$ is its complete sub-graph. A Co-Clique (independent set) in a graph is a set of pairwise non-adjacent vertices. The clique number $w(\xi)$ of a graph $\xi$ is the number of vertices in a maximum clique in $\xi$.

## 2.2 Theorem

Let S be a commutative semi ring with non-zero identity and suppose that J denotes the Jacobson radical of S and $\xi(\frac{S}{J})$ is semiunit graph of $\frac{S}{J}$
(a) If x,y $\in S$ and 2x $\notin S_u$ then x + J is a co-clique in $\xi(S)$ .
(b) If x,y $\in S$ and 2x $\in S_u$ then x + J is a clique in $\xi(S)$ .
**Proof**
(a) Consider 2x $\notin S_u$ and suppose on contrary that $z_1, z_2 \in x+J$ such that $z_1, z_2$ and are adjacent in $\xi(\frac{S}{J})$ , therefore, $z_1+z_2 \in (\frac{S}{J})_u$ , semi units of $\frac{S}{J}$.Where $z_1=x+j_1, z_2=x+j_2$ , $j_1, j_2 \in J$ .Therefore, $z_1+z_2=2x+j_1+j_2 \in S_u$. By assumption, 2x $\notin S_u$ . So, there exist maximal ideal M of S such that,

$$\langle 2x\rangle \subseteq M \Longrightarrow 2x \in M \qquad (1)$$

Also $j_1, j_2 \in J \subseteq M \Longrightarrow 2x+j_1+j_2 \in M \Longrightarrow z_1+z_2 \in M$ by (1)
which is a contradiction,so x + J is co-clique in $\xi$ .

(b) Consider 2x $\in S_u$ and suppose that $z_1, z_2 \in x+J$ such that $z_1, z_2$ and are non-adjacent in $\xi(\frac{S}{J})$ ,Where $z_1=x+j_1, z_2=x+j_2$ , $j_1, j_2 \in J$ ,therefore $z_1+z_2$ is non semi unit of S therefore there exist a k-maximal ideal M (by Proposition 2.1) such that $\langle z_1+z_2\rangle \subseteq M$ then

$$z_1+z_2=2x+j_1+j_2 \in M \Longrightarrow 2x \in M \text{ , since } j_1+j_2 \in M$$

which is contradiction as 2x is semiunit, so x + J is clique in $\xi$.

## 2.3 Theorem

If S is finite partitioning semi ring and J(S) is Q-ideal such that $|Q| = n$. If $\xi(\frac{S}{J})$ is complete and $2 \notin S_u$, then $\xi(Q \oplus J)$ is complete n-partite .

**Proof** Here, $Q \oplus J = \{q + j, \forall q \in Q, j \in J\}$ . Given that, $\xi(\frac{S}{J})$ is complete that is every element x + J, y + J are adjacent for all x, y $\in Q$ . Then every element of x + J are adjacent with every element of y + J (by Theorem [4.1]). We only need to show that elements of x + J are not adjacent with each others. For this consider, r, s $\in x + J$ such that $r = x + j_1$ *and* $s = x + j_2$

$$\Longrightarrow r + s = 2x + j_3 \qquad (1) \text{ ,where } j_3 = j_1 + j_2.$$

Also we have $\xi(\frac{S}{J})$ is complete which tells that $(\frac{S}{J})$ is semifield with char$((\frac{S}{J}))$ = 2 (by Theorem [3.13]). This tells that J is maximal ideal therefore
$2 \notin S_u \Rightarrow 2 \in J$, that is 2x $\in J$

$$r + s = 2x + j_3 \in J$$

Hence r,s are non-adjacent so $\xi(Q \oplus J)$ is complete n-partite .

## 2.4 Theorem

If S is commutative local semi ring with non-zero identity, J is its jacobson radical. If J is Q-ideal also, then f : $S_u \longrightarrow (\frac{S}{J})^*$ is an isomorphism, here $(\frac{S}{J})^*$ is set of all Q-cosets except 0 + J

**Proof** By defining the mapping

$$f(r + a) = r + J, where 0 \neq r \in Q \textit{ and } a \in J$$

we can easily check that r + a is semi unit and the function f is isomorphism.

### 2.4.1 Corollary

If S is commutative local semi ring with non-zero identity, J is its jacobson radical. If J is Q-ideal also, then $\xi(S_u)$ is isomorphic graph to $\xi(\frac{S}{J})^*$

## 2.5 Theorem

Let I be a proper Q-ideal of a semi ring S. Then $gr(\xi(S)) \leq gr(\xi(\frac{S}{I}))$. In particular, if $\xi(\frac{S}{I})$ contains a cycle, then so does $\xi(S)$.

**Proof** We may assume that there exist a cycle in $\frac{S}{I}$ with $gr(\frac{S}{I}) = n < \infty$. Let$x_1+I$ — $x_2+I$— .....— $x_n+I$ be a cycle in $\xi(\frac{S}{I})$ through n distinct vertices of $\frac{S}{I}$, where $x_i \in Q f$or $1 \leq$ i $\leq$ n. Then there must exist atleast this cycle $x_1$ — $x_2$ — ......—$x_n$ in $\xi(S)$ of length n, since $x_1 + x_2, x_2 + x_3, ...., x_{n-1} + x_n$ are semi units. ■

A vertex x of a connected graph $\xi$ is a cut-point of $\xi$ if there are vertices y,z of $\xi$ such that x is in every path from y to z $(and x \neq y, x \neq z)$. Equivalently, for a connected graph $\xi$, x is a cut-point of $\xi$ $if$ $\xi - \{x\}$ is not connected

## 2.6 Lemma [11]

Let S be a semiring and let r $\in$ S Then:
(i) If r is a nilpotent element of S, then it is not a semi-unit.
(ii) If r $\in$ Jac(S), then for every a $\in$ S the element 1 + ra is a semi-unit of S.

## 2.7 Proposition

If S is a semi ring with identity and J is Jacobson radical , then in quotient S/J ,
( a ) the coset 1 + J is always connected with q + J whenever $q \in r.J, r \in S$ in $\xi(\frac{S}{J})$.
( b ) the set J U {1} is always connected in $\xi$(S) which is a star subgraph with 1 is the cut point of this connected subgraph in $\xi$(S).
( c ) If N is Nil radical , then in quotient S/N ,the coset 0 + N = N is always null sub graph in $\xi$(S)

**Proof**

(a) Obviously, the cosets 1 + J and q + J , where q $\in r.J$, are adjacent
(b) If r $\in$ Jac(S), then for every a $\in$S, the element 1 + ra is a semi-unit of S, so if 1 +ra is semi unit for all a $\in$S implies that 1 + 1.r $\in S_u$.We have the subset J U {1} from S than clearly this would be connected sbgraph in $\xi$(S) along with 1 is the cut point since if we delete 1 from this set then the graph becomes disconnected , also it makes the star graph.

(c) If r is a nilpotent element of S, then it is not a semi-unit by (Lemma 4.6) and Nil radical contains all nilpotent elements, therefore set 0 + N makes empty subgraph in $\xi$ (S).

### 2.7.1 Corollary

If S is a local semi ring with identity and J is Jacobson radical , then in quotient S/J ,
( a ) the set J U {1} is always connected in $\xi$(S) which is biggest star subgraph with 1 is the cut point of this connected subgraph in $\xi$(S).
( b ) If N is Nil radical , then in quotient S/N ,the coset 0 + N = N is co-clique in $\xi$(S).

## 2.8 Proposition

If S is semi field then $\xi(S) \cong \xi(\frac{S}{J})$

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